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Thème 4 — Simulation et optimisation
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Résumé : Nous considérons un système faiblement couplé d'équations semi-linéaires de type hyperbolique-parabolique avec une diffusion anisotrope dégénérée. Ce système se rencontre dans la modélisation de l'évolution d'un traceur chimique ou biologique dans un milieu poreux. Nous étudions ce problème avec une théorie L^1, puis nous établissons la limite quand la constante de réaction devient grande. Nous montrons que le système converge vers une équation non linéaire parabolique-hyperbolique qui généralise le problème de Stefan. Deux spécificités de cet article sont (i) le traitement de données initiales "mal préparées" et (ii) l'unicité des solutions entropiques basée sur une inégalité d'entropie précise.

Mots-clés : Transport et réaction en milieux poreux, Système hyperbolique-parabolique dégénéré, Problème de Stefan généralisé.

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A multiscale colloid transport model with anistropic degenerate diffusion

Abstract: We consider a weakly coupled semilinear parabolic-hyperbolic system with a degenerate and anisotropic diffusion. It arises to model the evolution of a chemical or biological tracer in a porous medium. We study the well-posedness of the system using a L^1 theory. Then, we establish the relaxation limit as the reaction constant becomes large. We prove that the system converges to a nonlinear parabolic-hyperbolic equation that generalizes the Stefan problem. Two specificities of this paper are (i) to deal with ill-prepared initial data and (ii) with unique entropy solutions based on a precise entropy inequality.

Key-words: Transport and reaction in porous media, Degenerate parabolic-hyperbolic system, generalized Stefan problem.

1 Introduction

This note deals with the following problem: find the solution c and s defined on $\mathbb{R}^d \times (0,T)$ (with $T>0$ and $d \leq 3$), to the weakly coupled semilinear and degenerate parabolic-hyperbolic system

$$(P_k)\begin{cases} \partial_t c + \mathrm{div}\mathbf{A}(c) - \displaystyle\sum_{i,j=1}^{d} \frac{\partial^2 \phi_{ij}(c)}{\partial x_i \partial x_j} = -k\,c\,s \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}^d), \\ \partial_t s = -k\,c\,s \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}^d), \\ c(0,x) = c_0(x) \geq 0 \quad \text{a.e. in } \mathbb{R}^d, \\ s(0,x) = s_0(x) \geq 0 \quad \text{a.e. in } \mathbb{R}^d, \end{cases} \tag{1}$$

where k is a given positive constant. The assumptions on the data $\mathbf{A} : \mathbb{R} \to \mathbb{R}^d$, the $d \times d$ matrix $\phi(c) = (\phi_{ij}(c))$, c_0 and s_0 will be made precise later (Section 2), but let us just say that they cover the cases of completely degenerated diffusion (we only assume (ϕ'_{ij}) is a nonnegative matrix) and ill-prepared initial data (the product $c_0\, s_0$ does not vanish). Our purpose is to prove a global well-posedness theory for this system and to study the relaxation limit as $k \to \infty$, and especially to characterize this limit through the generalized Stefan equation

$$(Q_\infty)\begin{cases} \partial_t w + \mathrm{div}\mathbf{A}(w_+) - \displaystyle\sum_{i,j=1}^{d} \frac{\partial^2 \phi_{ij}(w_+)}{\partial x_i \partial x_j} = 0 \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}^d), \\ w(0,x) = c_0(x) - s_0(x) \quad \text{a.e. in } \mathbb{R}^d. \end{cases} \tag{2}$$

One of the difficulties is that the parabolic degeneracy leads to singular solutions (possibly shock waves in the purely hyperbolic case). In order to cover this generality, we therefore use methods based on S. Kruzhkov's style L^1 contraction property [12].
The Figure 1 shows the values of c and s obtained numerically in $1D$, with $\mathbf{A}(c) = u\,c$ and $\phi(c) = \mu\, c$. The numerical values used for this simulation are: $k = 10^4$, $\mu = 4\,10^{-3}$ and $u = 0.1$. We notice the presence of propagation front across which s presents a discontinuity.

This kind of system describes the evolution of a *tracer* (typically a chemical or a biological species) in a porous medium. This tracer is assumed to adhere to the surface of the solid skeleton. The places where this adhesion process occurs are named the *adsorption sites.* Denoting by C the mobile tracer, by S the adsorption sites on the immobile porous medium, and $\overline{CS}$ the product of the reaction between the tracer and the skeleton, this process can be represented as a formal chemical reaction:

$$C + S \rightleftharpoons \overline{CS}.$$

We let c and s denote the concentration of C and S respectively. The concentrations c and s are classically assumed to satisfy the following system:

$$\begin{cases} \partial_t c + \mathrm{div}\mathbf{A}(c) - \displaystyle\sum_{i,j=1}^{d} \frac{\partial^2 \phi_{ij}(c)}{\partial x_i \partial x_j} = -R(c,s), \\ \partial_t s = -R(c,s), \end{cases}$$

Where the function R is a reaction term. In this work, we assume that the above pseudo-chemical reaction is governed by the law of mass action and we neglect the backward reaction (namely the desorption). Thus, we have

$$R(c,s) = k\,c\,s, \tag{3}$$

where k is the forward rate constant.

This model is encountered in various applications. For example, it was proposed in cancer research to study the penetration of antibodies in tumurous tissue and their attachment to antigens (K. Fenmori *et al.* [7]). It is also used to study the transport and attachment of colloids, bacteria or viruses, in sandy aquifers. In these later cases the function $R(c,s)$ generally takes the form $cB(s)$, where B, the so-called *blocking function*, is typically polynomial (see for example N. Sun *et al.* [17], C.H. Bolster *et al.* [2]). These equations can also be viewed as a simple model of colloid filtration, when the retained particules are small enough to consider that their adhesion on the filter do not modify the size of the pores (see M. Belhadj [1] for an application to blood filtration).

Concerning the mathematical theory, D. Hilhorst, R. van der Hout and L.A. Peletier proposed various mathematical studies on systems of this kind. In [8], they considered the large time behaviour and the limiting behaviour as $k \to \infty$ of a 1D version of this problem with $\mathbf{A} = 0$ and $\phi(c) = c$. In [9], they considered also the 1D case with $\mathbf{A} = 0$ and $\phi(c) = c$, but they took more general reaction terms. In [10], they kept $\mathbf{A} = 0$ but they extended their works to the multidimensional cases and nonlinear diffusions. More precisely they assume that $\phi(c) = \int_0^c D(\xi)\,d\xi$ with $D(\xi) > 0$ for $\xi > 0$. On the other hand, the purely hyperbolic case $\phi(c) = 0$ has been widely studied and we refer to the survey paper of R. Natalini [15] for further references and analysis. We would like also to point out that a possible approach is to use the nonlinear semigroup theory which provides easily a partial answer (uniqueness in a restricted sense) to the problem in the isotropic case at least.

In the present work, we restrict ourselves to the mass action kinetics (3), which is just one of the two kinds of reactions investigated in [10], but we consider a more general equation since we treat a completely degenerated and non-isotropic diffusions. Moreover, we also add the nonlinear convection term $\mathrm{div}\mathbf{A}(c)$, thus dealing with the complete hyperbolic-parabolic case. Finally, we do not assume that initial data are well-prepared i.e. $c_0 s_0 = 0$ and thus we implicitely treat the initial layer problem: a non-equilibrium initial data (an equilibrium is defined by a state (c,s) such that $c\,s = 0$) is immediately relaxed to an equilibrium in the limit $k \to \infty$. As mentioned earlier, we consider a purely L^1 well posedness theory. This requires to use a precise definition of entropy solutions, an issue solved in the purely hyperbolic case by S. Kruzhkov [12]. The parabolic case is much more involved and the entropy dissipation has to be controlled precisely in the relaxation process and in the construction of the solution. The main issue then is uniqueness of solutions satisfying these entropy inequalities, a result proved recently for isotropic matrices ($\phi_{ij} = 0 \quad \forall\, i \neq j$) by J. Carrillo [4] (see also R. Eymard *et al.* [6] for improved assumptions and K. H. Karlsen and N. H. Risebro [11] for numerical analysis of the problem). The recent improvment in this uniqueness proof by G. Q. Chen and B. Perthame [5] allows to treat non-isotropic degenerate diffusions to the expense of more precise entropy inequalities.

The paper is organized as follows. In Section 2, we prove an existence result for the above system based on the L^1 contraction property. We study the behaviour of the solution as k tends to infinity in Section 3.

Remarks on notations We use the standard notations for the Sobolev spaces, but for simplicity, we will sometimes denote the spaces $L^1((0,T)\times\mathbb{R}^d)$ by $L^1_{t,x}$, $L^2(0,T;H^1(\mathbb{R}^d))$ by $L^2_t(H^1_x)$, and so on. Moreover, in the sequel C will denote various constants.

2 Existence result

We make the following assumptions on the data:

$$(\phi_{ij})\in\mathcal{C}^2(\mathbb{R})\text{ is symmetric, }(\phi'_{ij})\text{is a nonnegative matrix,} \tag{4}$$

$$\mathbf{A}=(A_1,\dots,A_d)\in(\mathcal{C}^1(\mathbb{R}))^d, \mathbf{A}(0)=0, \tag{5}$$

$$c_0\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d), c_0\geq 0 \text{ a.e. in } \mathbb{R}^d, \text{ and } \int_{\mathbb{R}^d}|x|^2 c_0\,dx<\infty, \tag{6}$$

$$s_0\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d), \text{ and } s_0\geq 0 \text{ a.e. in } \mathbb{R}^d. \tag{7}$$

We denote by $(\phi')^{1/2}$ a symmetric square root of ϕ' and ψ its antiderivative,

$$\sum_{k=1}^{d}(\phi'^{1/2})_{ik}(\phi'^{1/2})_{jk}=\phi'_{ij}\,,\quad \psi'_{ik}(c)=(\phi'^{1/2}(c))_{ik}. \tag{8}$$

The main result of this Section is the following existence result which states the main properties that will be used for the relaxation limit.

Theorem 1
Under assumptions (4)-(7), problem (P_k) admits a unique solution $(c,s)\in\mathcal{C}(\mathbb{R}^+;L^1(\mathbb{R}^d))^2$. Moreover the following properties hold:

(i) Maximum principle:

$$0\leq c(t,x)\leq\|c_0\|_{L^\infty(\mathbb{R}^d)}, \qquad \text{a.e. } in\ \mathbb{R}^+\times\mathbb{R}^d, \tag{9}$$

$$0\leq s(t,x)\leq\|s_0\|_{L^\infty(\mathbb{R}^d)}, \qquad \text{a.e. } in\ \mathbb{R}^+\times\mathbb{R}^d. \tag{10}$$

(ii) Contraction property: let (c,s) and $(\overline{c},\overline{s})$ be two solutions corresponding respectively to the initial data (c_0,s_0) and $(\overline{c}_0,\overline{s}_0)$. Then for any $t\geq 0$,

$$\|c(t)-\overline{c}(t)\|_{L^1(\mathbb{R}^d)}+\|s(t)-\overline{s}(t)\|_{L^1(\mathbb{R}^d)}\leq\|c_0-\overline{c}_0\|_{L^1(\mathbb{R}^d)}+\|s_0-\overline{s}_0\|_{L^1(\mathbb{R}^d)}. \tag{11}$$

(iii) Comparison property: let (c,s) and $(\overline{c},\overline{s})$ be two solutions corresponding respectively to the initial data (c_0,s_0) and $(\overline{c}_0,\overline{s}_0)$. If $c_0\leq\overline{c}_0$ and $s_0\leq\overline{s}_0$ then for any $t\geq 0$,

$$c(t)\leq\overline{c}(t)\quad \text{and}\quad s(t)\leq\overline{s}(t)\quad \text{a.e. } in\ \mathbb{R}^d. \tag{12}$$

(iv) Entropy inequality: for any two smooth increasing functions S and Σ , with S convex, and with the notations $(\mathbf{A}^{\mathbf{S}})' = \mathbf{A}'S'$, $(\phi^S_{ij})' = \phi'_{ij}\, S'$, $(\psi^S_{ik})' = (S'')^{1/2}\,(\phi'^{1/2})_{ik}$, we have $\forall\, k = 1, \dots d$,

$$\sum_{i=1}^{d} \frac{\partial(\phi'^{1/2}(c))_{ik}}{\partial x_i} \in L^2(\mathbb{R}^+ \times \mathbb{R}^d),$$

$$S''(c)\sum_{i=1}^{d} \frac{\partial(\phi'^{1/2}(c))_{ik}}{\partial x_i} = \sum_{i=1}^{d} \frac{\partial \psi^S_{ik}(c)}{\partial x_i} := (\nabla \cdot \psi^S(c))_k,$$

$$\partial_t[S(c) + \Sigma(s)] + div\mathbf{A}^{\mathbf{S}}(c) - \sum_{i,j=1}^{d} \frac{\partial^2 \phi^S_{ij}(c)}{\partial x_i \partial x_j} + \sum_{k=1}^{d} |\nabla \cdot \psi^S(c)|^2 \leq 0. \tag{13}$$

Before recalling the proof of this Theorem, let us point out that, even though we did not see a complete statement and proof for a coupled system, the principles behind are not new (see for instance [4], [5], [6] or the earlier work by P. Marcati [14]). Also some assumptions can be slightly improved; the space quadratic moment in (6) is not needed, and a purely L^1 assumption on c is enough (see [16], [5]).

Lemma 1
We assume there is an $\alpha > 0$ such that

$$(\phi') \geq \alpha Id. \tag{14}$$

Then any solution $(c, s) \in \mathcal{C}^\infty(\mathbb{R}^+; \mathcal{S}(\mathbb{R}^d))$ to equations (1) i.e. with data in $\mathcal{S}(\mathbb{R}^d)$, satisfies properties (9)–(13). In addition we have for any $t \geq 0$,

$$\begin{aligned} \|\partial_t c(t)\|_{L^1(\mathbb{R}^d)} + \|\partial_t s(t)\|_{L^1(\mathbb{R}^d)} \quad &\leq \quad \beta \|\nabla c_0\|_{L^1(\mathbb{R}^d)} + \|\sum_{i,j=1}^{d} \frac{\partial^2 \phi_{ij}(c_0)}{\partial x_i \partial x_j}\|_{L^1(\mathbb{R}^d)} \\ &\qquad + 2k\|c_0 s_0\|_{L^1(\mathbb{R}^d)} \end{aligned} \tag{15}$$

where $\beta = \sup_{|\xi| \leq \|c_0\|_{L^\infty(\mathbb{R}^d)}} |\mathbf{A}'(\xi)|$.

Proof of Lemma 1. We first notice that

$$s(t,x) = s_0(x) \exp\left(-k \int_0^t c(\xi, x)\, d\xi\right), \tag{16}$$

thus, in view of (7), $s \geq 0$ almost everywhere. Multiplying the first equation of (1) by $c_- = \max(0, -c)$ and integrating, we obtain:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} c_-^2 + \int_{\mathbb{R}^d} |\phi'^{1/2}(c) \cdot \nabla c_-|^2 = -\int_{\mathbb{R}^d} ks(c_-)^2 - \int_{\mathbb{R}^d} \mathbf{A}'(c) \cdot \nabla c_- c_-.$$

For the time being, we assume there exists $\beta > 0$ such that

$$\|A_i'\|_{L^\infty(\mathbb{R})} \leq \beta, \tag{17}$$

for $i = 1, \ldots, d$. Thus,

$$\frac{d}{dt}\int_{\mathbb{R}^d} c_-^2 + \alpha \int_{\mathbb{R}^d} |\nabla c_-|^2 \leq \frac{\beta^2}{\alpha}\int_{\mathbb{R}^d} c_-^2.$$

Gronwall's lemma thus gives $c_- = 0$ since c_0 is supposed to be nonnegative. Next, introducing the function $v = c - \|c_0\|_{L^\infty(\mathbb{R}^d)}$ and using analogous argument with the function $v_+ = \max(0, v)$, we finally obtain (9). Notice that relation (10) is obvious in view of (16).

Next, we relax assumption (17) by a standard argument that we just sketch: we introduce a cut-off function $\zeta \in \mathcal{C}^\infty(\mathbb{R})$ satisfying $\zeta(\xi) = 1$ if $\xi \leq \|c_0\|_{L^\infty(\mathbb{R}^d)}$ and $\zeta(\xi) = 0$ if $\xi \geq \|c_0\|_{L^\infty(\mathbb{R}^d)} + 1$, and we let $\tilde{\mathbf{A}} = \zeta\mathbf{A}$. The function $\tilde{\mathbf{A}}$ clearly satisfies a relation like (17), thus we can use the above arguments. In particular we can prove (9), which implies $\tilde{\mathbf{A}}(c) = \mathbf{A}(c)$.

We now turn to the proof of properties (11) and (12). Let S_δ be a $\mathcal{C}^2(\mathbb{R})$ convex real function, and let (c, s) and $(\overline{c}, \overline{s})$ be two regular solutions to (1) corresponding to the initial regular data (c_0, s_0) and $(\overline{c}_0, \overline{s}_0)$. A substraction and a multiplication by $S_\delta'(c - \overline{c})$ yields:

$$\begin{aligned}\frac{\partial}{\partial t}S_\delta(c-\bar{c}) + \operatorname{div}(S_\delta'(c-\bar{c})(\mathbf{A}(c) - \mathbf{A}(\bar{c}))) - S_\delta''(c-\bar{c})\nabla(c-\bar{c})\cdot(\mathbf{A}(c) - \mathbf{A}(\bar{c}))\\ -S_\delta'(c-\bar{c})\sum_{i,j=1}^{d}\frac{\partial^2(\phi_{ij}(c) - \phi_{ij}(\bar{c}))}{\partial x_i \partial x_j} = S_\delta'(c-\bar{c})(-k(c-\bar{c})s + k\bar{c}(s-\bar{s})).\end{aligned} \tag{18}$$

The integral over $\mathbb{R}^d$ of the second term of (18) vanishes. We estimate the third one as follows:

$$-\int_{\mathbb{R}^d} |S_\delta''(c-\overline{c})\nabla(c-\bar{c})\cdot(\mathbf{A}(c) - \mathbf{A}(\bar{c}))| = \int_{\mathbb{R}^d} |(c-\overline{c})S_\delta''(c-\overline{c})\nabla(c-\bar{c})\cdot\frac{\mathbf{A}(c) - \mathbf{A}(\overline{c})}{c-\overline{c}}|.$$

Thus, we can choose S_δ such that $S_\delta(c) \to |c|$ in $L^\infty(\mathbb{R})$ as $\delta \to 0$, and $(c-\overline{c})S_\delta''(c-\overline{c})$ bounded and vanishing as $\delta \to 0$ uniformly away from 0. Therefore the third term of (18) also vanishes in $L^1(\mathbb{R}^d)$ for all times because c, $\overline{c}$ are smooth and decaying here. For the fourth term, we notice that:

$$
\begin{aligned}
&-\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} S'_\delta(c-\overline{c})\frac{\partial^2\Big(\phi_{ij}(c)-\phi_{ij}(\overline{c})\Big)}{\partial x_i\partial x_j} = \sum_{i,j=1}^{d}\int_{\mathbb{R}^d} S''_\delta(c-\overline{c})\partial_{x_i}(c-\overline{c})\Big(\phi'_{ij}(c)\partial_{x_j}c-\phi'_{ij}(\overline{c})\partial_{x_j}\overline{c}\Big)\\
&=\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} S''_\delta(c-\overline{c})\partial_{x_i}(c-\overline{c})\phi'_{ij}(c)\partial_{x_j}(c-\overline{c})+\\
&\quad\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} S''_\delta(c-\overline{c})\partial_{x_i}(c-\overline{c})\Big(\phi'_{ij}(c)-\phi'_{ij}(\overline{c})\Big)\partial_{x_j}\overline{c}\\
&=\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} S''_\delta(c-\overline{c})\partial_{x_i}(c-\overline{c})\phi'_{ij}(c)\partial_{x_j}(c-\overline{c})+\\
&\quad\sum_{i,j=1}^{d}\int_{\mathbb{R}^d}(c-\overline{c})S''_\delta(c-\overline{c})\partial_{x_i}(c-\overline{c})\frac{\phi'_{ij}(c)-\phi'_{ij}(\overline{c})}{c-\overline{c}}\partial_{x_j}\overline{c}.
\end{aligned}
$$

The first term is nonnegative. Passing to the limit as $\delta\to 0$ in the second, it vanishes as in the third term.
Finally, we deduce from (18)

$$\frac{d}{dt}\int_{\mathbb{R}^d}|c-\overline{c}| \leq -k\int_{\mathbb{R}^d}|c-\overline{c}|s + k\int_{\mathbb{R}^d}\overline{c}|s-\overline{s}|.$$

Similarly, we easily obtain:

$$\frac{d}{dt}\int_{\mathbb{R}^d}|s-\overline{s}| \leq k\int_{\mathbb{R}^d}|c-\overline{c}|s - k\int_{\mathbb{R}^d}\overline{c}|s-\overline{s}|.$$

By summing these last two inequalities, we deduce

$$\frac{d}{dt}\int_{\mathbb{R}^d}(|c-\overline{c}|+|s-\overline{s}|)\leq 0,$$

which gives the contraction property (11). Choosing S_δ such that $S_\delta(c)\to c_+$ as $\delta\to 0$ gives, by the same arguments, the comparison property (12).

Finally, let us prove (15). We denote $\partial_t c$ (resp. $\partial_t s$) by u (resp. v), we differentiate the first two equations of (1) with respect to t and we multiply them respectively by $\mathrm{sgn}(u)$ and $\mathrm{sgn}(v)$:

$$
\left\{
\begin{aligned}
\frac{\partial|u|}{\partial t}+\sum_{i=1}^{d}\frac{\partial}{\partial x_i}\Big(A'_i(c)|u|\Big)-\mathrm{sgn}(u)\sum_{i,j=1}^{d}\frac{\partial^2\Big(\phi'_{ij}(c)u\Big)}{\partial x_i\partial x_j} &= -k|u|s-\mathrm{sgn}(u)kcv,\\
\partial_t|v| &= -\mathrm{sgn}(v)kus-kc|v|.
\end{aligned}
\right.
\tag{19}
$$

Thanks to the decay assumption, $(c,s) \in \mathcal{S}(\mathbb{R}^d)$, the integral of $\partial_{x_i}\Big(A_i'(c)|u|\Big)$ over $\mathbb{R}^d$ vanishes. Let us check that the integral of $-\mathrm{sgn}(u)\displaystyle\sum_{i,j=1}^{d} \frac{\partial^2\Big(\phi_{ij}'(c)u\Big)}{\partial x_i \partial x_j}$ is nonnegative. We denote by H_δ a monotone C^∞ function such that $H_\delta(u) \to \mathrm{sgn}(u)$ in $L^1(\mathbb{R})$ as $\delta \to 0$, we have:

$$\begin{aligned} -\int_{\mathbb{R}^d} H_\delta(u) \frac{\partial^2\Big(\phi_{ij}'(c)u\Big)}{\partial x_i \partial x_j} &= \int_{\mathbb{R}^d} H_\delta'(u)\partial_{x_i} u \Big(\phi_{ij}'(c)\partial_{x_j} u + \partial_{x_j}\phi_{ij}'(c)u\Big) \\ &\geq \int_{\mathbb{R}^d} \frac{\partial}{\partial x_i} H_\delta(u) u \partial_{x_j}\phi_{ij}'(c) \\ &= -\int_{\mathbb{R}^d} \mathcal{K}_\delta(u) \frac{\partial^2 \phi_{ij}'(c)}{\partial x_i \partial x_j}. \end{aligned}$$

With $\mathcal{K}_\delta'(u) = H_\delta'(u)u$ and $\mathcal{K}_\delta(u) \to 0$ as $\delta \to 0$. Therefore letting $\delta \to 0$, this inequality becomes

$$-\int_{\mathbb{R}^d} \mathrm{sgn}(u) \frac{\partial^2\Big(\phi_{ij}'(c)u\Big)}{\partial x_i \partial x_j} \geq 0.$$

Thus, we deduce from (19),

$$\frac{d}{dt}\int_{\mathbb{R}^d} |u| + \frac{d}{dt}\int_{\mathbb{R}^d} |v| \leq -k\int_{\mathbb{R}^d} |u|s + k\int_{\mathbb{R}^d} c|v| + k\int_{\mathbb{R}^d} |u|s - k\int_{\mathbb{R}^d} c|v| \leq 0,$$

and therefore

$$\begin{aligned} \int_{\mathbb{R}^d} |\partial_t c| + |\partial_t s| &\leq \int_{\mathbb{R}^d} |(\partial_t c)_{t=0}| + |(\partial_t s)_{t=0}| \\ &\leq \int_{\mathbb{R}^d} |\mathrm{div}\mathbf{A}(c_0)| + \int_{\mathbb{R}^d} \Big|\sum_{i,j=1}^{d} \frac{\partial^2 \phi_{ij}(c_0)}{\partial x_i \partial x_j}\Big| + 2\int_{\mathbb{R}^d} k c_0 s_0. \end{aligned}$$

It remains to show (13), which simply follows from the chain rule and space-time integration (we may always assume that $S(0) = \Sigma(0) = 0$ and therefore S and Σ are nonnegative on the range of interest for (c,s)), thus we skip the calculations. This achieves the proof of Lemma 1. $\diamondsuit$

Lemma 2
We assume $c_0 \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, and we assume that the diffusion does not degenerate, i.e. *(14) is satisfied. Then, for all $T > 0$, problem (1) has a solution $(c,s) \in L^2(0,T;H^1(\mathbb{R}^d)) \cap \mathcal{C}([0,T];L^2(\mathbb{R}^d))$. If we assume moreover that $c_0, s_0 \in \mathcal{S}(\mathbb{R}^d)$ and $\mathbf{A}, \phi \in \mathcal{C}^\infty(\mathbb{R})$, then $c, s \in \mathcal{C}^\infty((0,T);\mathcal{S}(\mathbb{R}^d))$.*

Proof of Lemma 2.

Step 1. We first assume there exists $\beta > 0$ such that

$$\|\phi'\|_{L^\infty(\mathbb{R})} \leq \beta \quad \text{and} \quad \|A_i'\|_{L^\infty(\mathbb{R})} \leq \beta, \tag{20}$$

for $i = 1, \dots, d$. We restrict problem (1) to $\Omega_R = \{x \in \mathbb{R}^d, |x| \leq R\}$, R being a given nonnegative real, and we complement it with Dirichlet homogenous boundary conditions on $\partial\Omega_R$ for c. We thus consider the system

$$\partial_t c + \operatorname{div}\mathbf{A}(c) - \operatorname{div}(\phi'(c)\nabla c) = -k\,c\,s, \tag{21}$$

$$\begin{aligned}
\partial_t s &= -k\,c\,s, &(22)\\
c(t,x) &= 0 \text{ on } [0,T]\times\partial\Omega_R, &(23)\\
c(0,\cdot) &= c_0\chi_R \quad \text{on } \Omega_R, &(24)\\
s(0,\cdot) &= s_0\chi_R \quad \text{on } \Omega_R, &(25)
\end{aligned}$$

where χ_R denotes the characteristic function of Ω_R. We introduce the following convex set of the Banach space $L^2(0,T;L^2(\Omega_R))$:

$$\mathcal{C} = \{v \in L^2(0,T;L^2(\Omega_R)), \|v\|^2_{L^2(0,T;L^2(\Omega_R))} \leq m, \text{ and } v \geq 0 \text{ a.e. in } (0,T)\times\Omega_R\},$$

m being a nonnegative constant that will be fixed later. For $\overline{c} \in \mathcal{C}$, we define $\overline{s}$ by

$$\overline{s}(x,t) = s_0(x)\exp\left(-k\int_0^t \overline{c}(x,\xi)\,d\xi\right), \tag{26}$$

which is the solution to $\partial_t\overline{s} = -k\overline{c}\,\overline{s}$ satisfying $\overline{s}|_{t=0} = s_0$. Notice that in particular, for all $t \in [0,T]$, $\|\overline{s}(t)\|_{L^\infty(\Omega_R)} \leq \|s_0\|_{L^\infty(\Omega_R)}$. Then, we define c as the solution in $L^2(0,T;H_0^1(\Omega_R)) \cap L^\infty(0,T;L^2(\Omega_R))$ to:

$$\partial_t c + \operatorname{div}\left(\frac{\mathbf{A}(\overline{c})}{\overline{c}}c\right) - \operatorname{div}(\phi'(\overline{c})\nabla c) = -k\,c\,\overline{s}.$$

Notice that the existence of c comes from standard theory on non-degenerate linear parabolic problems. We denote by $\mathcal{T}$ the application $\overline{c} \to \mathcal{T}\overline{c} = c$. Our purpose is to prove that, for $\overline{c} \in \mathcal{C}$, $\mathcal{T}\overline{c} \in \mathcal{C}$ (for a suitable m), and to prove that $\mathcal{T}\overline{c}$ lies in a compact set of $L^2(0,T;L^2(\Omega_R))$ in order to apply Schauder fixed point theorem.

It is well-known that $\partial_t c \in L^2(0,T;H^{-1}(\Omega_R))$, which yields in particular $c \in \mathcal{C}([0,T];L^2(\Omega_R))$ and which makes rigorous all the manipulations we are going to do now. Multiplying by c and integrating we obtain:

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_R} c^2 + \sum_{i,j=1}^d \int_{\Omega_R} \phi'_{ij}(\overline{c})\nabla_i c \cdot \nabla_j c = -k\int_{\Omega_R} c^2\overline{s} + \int_{\Omega_R} \frac{\mathbf{A}(\overline{c})}{\overline{c}} c \cdot \nabla c,$$

which gives, using (14), (20) and the fact that $\overline{s} \geq 0$,

$$\frac{d}{dt}\int_{\Omega_R} c^2 + \alpha\int_{\Omega_R} |\nabla c|^2 + \leq \frac{\beta^2}{\alpha}\int_{\Omega_R} c^2.$$

In particular,

$$\sup_{t\in[0,T]} \|c(t)\|^2_{L^2(\Omega_R)} \leq \|c_0\|_{L^2(\Omega_R)} e^{\beta^2 T/\alpha}.$$

Thus, choosing $m = T\|c_0\|_{L^2(\mathbb{R}^d)} e^{\beta^2 T/\alpha}$, we have $\|c\|_{L^2(0,T;L^2(\Omega_R))} \leq m$. Next, multiplying (21) by $c_- = \max(0,-c)$ and integrating, we obtain:

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_R} c_-^2 + \int_{\Omega_R} \frac{\mathbf{A}'(\overline{c})}{\overline{c}} \cdot \nabla c_- c_- + \sum_{i,j=1}^{d} \int_{\Omega_R} \phi'_{ij}(\overline{c}) \nabla_i c \cdot \nabla_j c = -\int_{\Omega_R} ksc_-,$$

which gives:

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_R} c_-^2 + \alpha \int_{\Omega_R} |\nabla c_-|^2 \leq \beta \int_{\Omega_R} |\nabla c_-| c_-.$$

Young inequality and Gronwall's lemma thus give $c_- = 0$, since c_0 is supposed to be non-negative. Thus $c \in \mathcal{C}$.
We now check that $\partial_t c$ is bounded in $L^2(0,T;H^{-1}(\Omega_R))$. Let $v \in L^2(0,T;H^1_0(\Omega_R))$:

$$\begin{array}{rcl} \displaystyle\int_0^T < \partial_t c, v > &=& \displaystyle\int_0^T \int_{\Omega_R} \Big(-\phi'_{ij}(\overline{c}) \nabla_i c \cdot \nabla_j v + \frac{\mathbf{A}(\overline{c})}{\overline{c}} c \cdot \nabla v - k\overline{s}cv \Big) \\ &\leq& (\beta \|c\|_{L^2_t(H^1_x)} + \beta \|\overline{c}\|_{L^2_{x,t}}) \|v\|_{L^2_t(H^1_x)} + k\|s_0\|_{L^\infty_x} \|c\|_{L^2_{t,x}} \|v\|_{L^2_{t,x}} \\ &\leq& C\|v\|_{L^2_t(H^1_x)}. \end{array}$$

Thus, $\partial_t c$ is bounded in $L^2(0,T;H^{-1}(\Omega_R))$, and since c is also bounded in $L^2(0,T;H^1(\Omega_R))$, this proves that c is compact in $L^2(0,T;L^2(\Omega_R))$, and therefore the application $\mathcal{T}$ has a fixed point which is a solution to (21)-(25). Notice that we can finally relax assumption (20) as in the proof of Lemma 1.

Step 2. We denote by (c_R, s_R) the solution on Ω_R built above. For $R_1 > R_2$, we have obviously $c_0\chi_{R_1} \geq c_0\chi_{R_2}$ and $s_0\chi_{R_1} \geq s_0\chi_{R_2}$. Thus, for all $t > 0$, $(c_R(t), s_R(t))_R$ is an increasing sequence, bounded in $(L^2(\mathbb{R}^d))^2$ (in step 1, the definition of the bound m was independent of Ω_R). Letting $R \to \infty$ and applying the monotone convergence theorem, this proves the existence of the solution on $\mathbb{R}^d$. The regularity result is a consequence of classical theory of non-degenerate parabolic problems (see [13] and [18] for example). $\diamondsuit$

Proof of Theorem 1. We proceed in two steps.

Step 1. In this step, we assume that $c_0 \in \mathcal{S}(\mathbb{R}^d)$ and $s_0 \in \mathcal{S}(\mathbb{R}^d)$. We denote by ϕ_ε (resp. $\mathbf{A}_\varepsilon$) a suitable regularization of ϕ (resp. $\mathbf{A}$), in particular, we assume that $\phi'_\varepsilon \geq \varepsilon Id$. We consider the system

$$\left\{ \begin{array}{rcl} \partial_t c_\varepsilon + \mathrm{div}\mathbf{A}_\varepsilon(c_\varepsilon) - \mathrm{div}\Big(\phi'_\varepsilon(c_\varepsilon)\nabla c_\varepsilon\Big) &=& -k\, c_\varepsilon\, s_\varepsilon, \\ \partial_t s_\varepsilon &=& -k\, c_\varepsilon\, s_\varepsilon. \end{array} \right. \qquad (27)$$

We now aim at showing that the set $\{(c_\varepsilon, s_\varepsilon)\}_{\varepsilon>0}$ is compact in $\mathcal{C}(0,T;L^1(\mathbb{R}^d))$.

Let $h=(h_1,\dots,h_d)\in\mathbb{R}^d$, with $h_i>0$ for $i=1,\dots,d$. We define $\tau_h c_\varepsilon(t,x)=c_\varepsilon(t,x+h)$ and $\tau_h s_\varepsilon(t,x)=s_\varepsilon(t,x+h)$. Clearly, $\tau_h c_\varepsilon$ and $\tau_h s_\varepsilon$ satisfies (27) with initial data defined by $\tau_h c_0(x)=c_0(x+h)$ and $\tau_h s_0(x)=s_0(x+h)$. Thus, the contraction property (11) gives for all $t\in[0,T]$:

$$\begin{aligned}\|\tau_h c_\varepsilon(t)-c_\varepsilon(t)\|_{L^1(\mathbb{R}^d)} &+\|\tau_h s_\varepsilon(t)-s_\varepsilon(t)\|_{L^1(\mathbb{R}^d)}\\ &\le \|\tau_h c_0-c_0\|_{L^1(\mathbb{R}^d)}+\|\tau_h s_0-s_0\|_{L^1(\mathbb{R}^d)}.\end{aligned}\tag{28}$$

The right-hand side of this inequality tends to zero as $h\to 0$, thus, the Riez-Fréchet-Kolmogorov theorem (see [3] for example) implies the local relative compactness of $(c_\varepsilon(t,\cdot),s_\varepsilon(t,\cdot))$ in $(L^1(\mathbb{R}^d))^2$ for all $t\in[0,T]$. To show the global relative compactness, we multiply

$$\partial_t c_\varepsilon+\mathrm{div}\mathbf{A}_\varepsilon(c_\varepsilon)-\mathrm{div}\Big(\phi'_\varepsilon(c_\varepsilon)\nabla c\Big)=-kc_\varepsilon s_\varepsilon$$

by $|x|^2/2$ and we integrate by parts. We obtain, using $c_\varepsilon\ge 0$ and $s_\varepsilon\ge 0$,

$$\frac{d}{dt}\int_{\mathbb{R}^d}\frac{|x|^2}{2}c_\epsilon\,dx-\int_{\mathbb{R}^d}x\cdot\mathbf{A}_\varepsilon(c_\epsilon)\,dx-d\int_{\mathbb{R}^d}Tr\phi_\varepsilon(c_\epsilon)\,dx\le 0.$$

Now, denoting $\int_{\mathbb{R}^d}\frac{|x|^2}{2}c_\epsilon\,dx$ by $\mathcal{R}(t)$, and using (9),

$$\frac{d}{dt}\mathcal{R}(t)\le C\left(\int_{\mathbb{R}^d}|x|c_\epsilon\,dx+\int_{\mathbb{R}^d}c_\epsilon\,dx\right),$$

where, here and in the sequel, C denotes various constants independent of ε. Thus

$$\frac{d}{dt}\mathcal{R}(t)\le C(\mathcal{R}(t)+M(t)),$$

with $M(t)=\int_{\mathbb{R}^d}c_\epsilon\,dx$. Since $M(t)\le M(0)$, Gronwall lemma implies that $\mathcal{R}(t)$ is bounded by a value independent of ε as soon as $\mathcal{R}(0)$ is finite, which was assumed in (6). This yields the equiintegrability of $\{c_\varepsilon\}_\varepsilon$. Therefore, using analogous arguments for $\{s_\varepsilon\}_\varepsilon$, we deduce the relative compactness of $\{(c_\varepsilon,s_\varepsilon)\}$ in $(L^1(\mathbb{R}^d))^2$ for all $t\in[0,T]$.

Finally, we obtain from (15), for $0<t_1<t_2<T-\eta$,

$$\begin{aligned}&\sup_{t\in[t_1,t_2]}\|c_\varepsilon(t+\eta)-c_\varepsilon(t)\|_{L^1(\mathbb{R}^d)}+\sup_{t\in[t_1,t_2]}\|s_\varepsilon(t+\eta)-s_\varepsilon(t)\|_{L^1(\mathbb{R}^d)}\\ \le&\sup_{t\in[t_1,t_2]}\left\|\int_0^\eta\partial_t c_\varepsilon(t+\xi)\,d\xi\right\|_{L^1(\mathbb{R}^d)}+\sup_{t\in[t_1,t_2]}\left\|\int_0^\eta\partial_t s_\varepsilon(t+\xi)\,d\xi\right\|_{L^1(\mathbb{R}^d)}\\ &\le\beta\eta\|\nabla c_0\|_{L^1(\mathbb{R}^d)}+\eta\|\mathrm{div}\Big(\phi'_\varepsilon(c_0)\nabla c_0\Big)\|_{L^1(\mathbb{R}^d)}+2k\eta\|c_0\,s_0\|_{L^1(\mathbb{R}^d)}.\end{aligned}$$

Thus, by Ascoli theorem, $\{(c_\varepsilon,s_\varepsilon)\}_\varepsilon$ is relatively compact in $\mathcal{C}(0,T;L^1(\mathbb{R}^d))^2$. Therefore, up to an extraction, $(c_\varepsilon,s_\varepsilon)$ converges in $\mathcal{C}(0,T;L^1(\mathbb{R}^d))^2$ and a.e. to (c,s). Passing to the limit in (27), we deduce that (c,s) is a solution to (1). Moreover passing to the limit in the relations established in Lemma 1, this solution satisfies (9)– (13).

Step 2. We now relax the regularity assumptions on (c_0, s_0). Let c_0^n and s_0^n be two sequences of $\mathcal{S}(\mathbb{R}^d)$ converging to c_0 and s_0 in $L^1(\mathbb{R}^d)$. According to step 1, there exists a solution (c_n, s_n) to system (1) corresponding to the initial data (c_0^n, s_0^n) which satisfies the contraction property (11). This last point implies in particular that (c_n, s_n) is a Cauchy sequence in the Banach space $\mathcal{C}(0,T; L^1(\mathbb{R}^d))$. Thus, (c_n, s_n) converges in $\mathcal{C}(0,T; L^1(\mathbb{R}^d))$ to (c,s) which is solution to (1) and which satisfies properties (9)–(12). As for (13), we have a uniform $L^2(\mathbb{R}^+ \times \mathbb{R}^d)$ bound on $\nabla \cdot \phi'^{1/2}(c^n)$. Therefore, following [5] the strong limit $\psi^S(c)$ of $\psi^S(c^n)$ satisfies the chain rule by strong-weak convergence, and $|\nabla \cdot \psi^S(c)| \leq lim_{n\to\infty} |\nabla \cdot \psi^S(c^n)|$ as a weak L^2 limit. Hence we deduce the inequality (13) for (c,s). $\diamondsuit$

3 Asymptotic behaviour

Theorem 2
We make the assumptions (4)-(7). As k tends to infinity, the solution (c_k, s_k) to (P_k), has a limit in $L^1((0,T)\times\mathbb{R}^d)^2$ denoted by $(c,s) \in L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^d))^2$ that satisfies

$$(P_\infty) \left\{ \begin{array}{rcl} \partial_t(c-s) + div\mathbf{A}(c) - \displaystyle\sum_{i,j=1}^{d} \frac{\partial^2 \phi_{ij}(c)}{\partial x_i \partial x_j} & = & 0, \quad in\ \mathcal{D}'((0,T)\times\mathbb{R}^d), \\ c \geq 0,\ s \geq 0,\ c\,s & = & 0, \quad \text{a.e. } in\ \Omega \times (0,T), \\ c(0,x) - s(0,x) & = & c_0(x) - s_0(x) \quad \text{a.e. } in\ \Omega. \end{array} \right.$$

Moreover, (c,s) is the unique entropy solution to (P_∞) i.e. for any two smooth increasing functions S and Σ, with S convex, and with the notations $(\mathbf{A}^\mathbf{S})' = \mathbf{A}'S'$, $(\phi^S)' = \phi' S'$, $(\psi^S)' = (S'')^{1/2}\phi'^{1/2}$, it satisfies

$$\nabla \cdot \phi'^{1/2}(c) \in (L^2(\mathbb{R}^+ \times \mathbb{R}^d))^d, \quad \nabla \cdot \psi^S(c) = (S'')^{1/2}\phi'^{1/2}(c),$$

$$\partial_t[S(c) + \Sigma(s)] + div\mathbf{A}^\mathbf{S}(c) - \sum_{i,j=1}^{d} \frac{\partial^2 \phi_{ij}^S(c)}{\partial x_i \partial x_j} + \sum_{k=1}^{d} |\nabla \cdot \psi^S(c)|^2 \leq 0. \tag{29}$$

Finally, $w = c - s \in \mathcal{C}(\mathbb{R}^+; L^1(\mathbb{R}^d))$ is the unique entropy solution to the generalized Stefan equation (2).

Proof.

Relation with the generalized Stefan equation, uniqueness Before proving the hard core of this theorem, let us explain the relation between its limit (P_∞) and (Q_∞) in (2). From the sign condition on (c,s), and $c\,s = 0$, we can invert $(c,s) \to w = c - s$ as follows:

$$w \geq 0 \leftrightarrow c = w,\ s = 0, \quad \text{and} \quad w \leq 0 \leftrightarrow s = -w,\ c = 0.$$

This defines a lipschitz continuous inverse for c: $c = w_+$ and the two problems are thus equivalent. Also the entropy inequalities can be translated in terms of w. We firstly restrict our choice to $S(0) = \Sigma(0) = 0$ and extend S to negative values by $S(w) = S(w_+) + \Sigma(w_-)$; thus we reach all smooth functions $S(w)$ such that S is convex for $w \geq 0$ and $\mathrm{sgn} S'(w) = \mathrm{sgn} w$, and we have $\nabla \cdot \psi^S(w) \in L^2(\mathbb{R}^+ \times \mathbb{R}^d)$ and

$$\partial_t S(w) + \mathrm{div}\mathbf{A}^{\mathbf{S}}(w) - \sum_{i,j=1}^{d} \frac{\partial^2 \phi_{ij}^S(w)}{\partial x_i \partial x_j} + |\nabla \cdot \psi^S(w)|^2 \leq 0, \tag{30}$$

with A^S, ϕ^S, ψ_{ij}^S extended by 0 to negative values of their argument. Endowed with its initial value, this hyperbolic-parabolic admits a unique entropy solution in $\mathcal{C}(\mathbb{R}^+; L^1(\mathbb{R}^d))$ (see the references in the introduction). Therefore we have obtained the relation between the two problems and the uniqueness of the limit.

Existence of a limit For $h \in \mathbb{R}^d$, we denote by $\tau_h c_k$ (resp. $\tau_h s_k$) the function $(t,x) \to \tau_h c_k(t,x) = c_k(t, x+h)$ (resp. $(t,x) \to \tau_h s_k(t,x) = s_k(t,x+h)$). For $\eta \in \mathbb{R}$, we denote by $\mathcal{T}_\eta c_k$ the function $(t,x) \to \mathcal{T}_\eta c_k(t,x) = c_k(t+\eta, x)$). We denote by $\omega(\cdot)$ the initial L^1 modulus of continuity

$$\omega(h) = \sup_{|\bar{h}| \leq h} \|c^0(\cdot + \bar{h}) - c^0(\cdot)\|_{L^1(\mathbb{R}^d)} + \|s^0(\cdot + \bar{h}) - s^0(\cdot)\|_{L^1(\mathbb{R}^d)}.$$

We begin by proving the compactness of s_k. Integrating $\partial_t s_k = -k c_k s_k$ and using $s_k \geq 0$, we obtain for all $T > 0$,

$$k \int_0^T\!\!\int_{\mathbb{R}^d} c_k s_k \, dx\, dt = - \int_{\mathbb{R}^d} \Big(s_k(T,x) - s_k(0,x) \Big)\, dx \leq \int_{\mathbb{R}^d} s_0 \, dx \stackrel{\text{def}}{=} m_0. \tag{31}$$

Thus we have

$$k \|c_k\, s_k\|_{L^1(\mathbb{R}^+ \times \mathbb{R}^d)} \leq m_0, \qquad c_k s_k \to 0 \text{ in } L^1(\mathbb{R}^+ \times \mathbb{R}^d) \text{ as } k \to \infty. \tag{32}$$

Notice that this also shows that $\|\partial_t s_k\|_{L^1_{t,x}} \leq m_0$, and since

$$\begin{aligned} \|\tau_h s_k - s_k\|_{L^1(\mathbb{R}^+ \times \mathbb{R}^d)} &\leq T \|\tau_h c_0 - c_0\|_{L^1(\mathbb{R}^d)} + T \|\tau_h s_0 - s_0\|_{L^1(\mathbb{R}^d)} \\ &\leq T\, \omega(|h|), \end{aligned}$$

we deduce the local compactness of $(s_k)_k$ in $L^1((0,T) \times \mathbb{R}^d)$.

Let us now prove the compactness of $(c_k)_k$ relying on an improvment by A. E. Tzavaras [19] of a classical regularization argument. First of all, we have space compactness because using (11),

$$\begin{aligned} \|\tau_h c_k - c_k\|_{L^1((0,T) \times \mathbb{R}^d)} &\leq T \|\tau_h c_0 - c_0\|_{L^1(\mathbb{R}^d)} + T \|\tau_h s_0 - s_0\|_{L^1(\mathbb{R}^d)} \\ &\leq T\, \omega(|h|). \end{aligned} \tag{33}$$

Let ε be a positive constant that will be fixed later, and let $\rho_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^d)$ be a mollifier kernel vanishing outside the ball of radius ε centered in 0. Let t_1 and t_2 be such that $0 < t_1 < t_2 < T - \eta$, we have

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\mathbb{R}^d} |\mathcal{T}_\eta c_k - c_k| \quad &\leq \quad \int_{t_1}^{t_2}\int_{\mathbb{R}^d} |\mathcal{T}_\eta c_k - \mathcal{T}_\eta(c_k * \rho_\varepsilon)| + \int_{t_1}^{t_2}\int_{\mathbb{R}^d} |c_k * \rho_\varepsilon - c_k| \\ &\quad + \int_{t_1}^{t_2}\int_{\mathbb{R}^d} |\mathcal{T}_\eta(c_k * \rho_\varepsilon) - c_k * \rho_\varepsilon|.\end{aligned}$$

The first and the second terms of the right hand side are treated by the same way:

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^d} |c_k * \rho_\varepsilon - c_k| \leq \int_{t_1}^{t_2}\int_{|y|<\varepsilon}\int_{\mathbb{R}^d} |c_k(t, x-y) - c_k(t,x)| \leq T\,\omega(\varepsilon).$$

For the third term of the right hand side, we first notice that

$$\partial_t(c_k * \rho_\varepsilon) = -\sum_{i=1}^d A_i(c_k) * \frac{\partial \rho_\varepsilon}{\partial x_i} + \sum_{i,j=1}^d \phi_{ij}(c_k) * \frac{\partial^2 \rho_\varepsilon}{\partial x_i \partial x_j} - k(c_k s_k) * \rho_\varepsilon,$$

thus, using (31),

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\mathbb{R}^d} |\mathcal{T}_\eta(c_k * \rho_\varepsilon) - c_k * \rho_\varepsilon| \quad &\leq \quad \int_{t_1}^{t_2}\int_{\mathbb{R}^d}\int_{\xi=0}^{\eta} |\partial_t(c_k * \rho_\varepsilon)(t+\xi, x)| \\ &\leq \quad \frac{\eta}{\varepsilon}\|\mathbf{A}(c_k)\|_{L^1_{t,x}} + \frac{\eta}{\varepsilon^2}\|\phi(c_k)\|_{L^1_{t,x}} + \eta\|k c_k s_k\|_{L^1_{t,x}} \\ &\leq \quad C\left(\eta + \frac{\eta}{\varepsilon} + \frac{\eta}{\varepsilon^2}\right),\end{aligned}$$

where C is a constant independent of ε, η and k. Thus, choosing $\varepsilon = \eta^{1/3}$, there exists a constant $C > 0$, independent of k, such that

$$\|\mathcal{T}_\eta c_k - c_k\|_{L^1((t_1,t_2)\times\mathbb{R}^d)} \leq C\left(\eta^{1/3} + \omega(\eta^{1/3})\right). \tag{34}$$

We deduce from (33) and (34) the local compactness of $(c_k)_k$ in $L^1((0,T)\times\mathbb{R}^d)$. We can check the equiintegrability of $(c_k)_k$ and $(s_k)_k$ as in the proof of Theorem 1. We deduce that the set $(c_k, s_k)_k$ is compact in $L^1((0,T)\times\mathbb{R}^d)$, and thus, up to an extraction, $(c_k, s_k) \to (c, s)$ in $L^1((0,T)\times\mathbb{R}^d)$ as $k \to \infty$. Passing to the limit in (1) and using (32), we finally obtain (P_∞).

Time continuity of w=c-s The above proof uses compactnes in $L^1((0,T)\times\mathbb{R}^d)$ in order to be compatible with the generic initial layer (and possibly shock layer) which leads to the fact that $\partial_t c$ and $\partial_t s$ are only bounded measures in time. Nevertheless an additional cancellation arises on $c-s$ which allows to prove time continuity. We argue as in previous subsection.

$$\begin{aligned}\int_{\mathbb{R}^d} |\mathcal{T}_\eta(c_k-s_k)*\rho_\varepsilon-(c_k-s_k)*\rho_\varepsilon| &\leq \int_{\xi=0}^{\eta}\int_{\mathbb{R}^d}|\partial_t(c_k-s_k)(\xi,\cdot)*\rho_\varepsilon| \\ &\leq \frac{\eta}{\varepsilon}\|\mathbf{A}(c_k)\|_{L^\infty_t L^1_x}+\frac{\eta}{\varepsilon^2}\|\phi(c_k)\|_{L^\infty_t L^1_x} \\ &\leq C\left(\frac{\eta}{\varepsilon}+\frac{\eta}{\varepsilon^2}\right).\end{aligned}$$

Also, we have

$$\int_{\mathbb{R}^d}|(c_k-s_k)*\rho_\varepsilon-(c_k-s_k)|\leq\omega(\varepsilon).$$

Therefore, we obtain choosing again $\varepsilon=\eta^{1/3}$, that, for some constant C independent of k and η, we have

$$\int_{\mathbb{R}^d}|\mathcal{T}_\eta(c_k-s_k)-(c_k-s_k)|\leq C\left(\eta^{1/3}+\omega(\eta^{1/3})\right).$$

which proves time continuity uniformly in k and achieves the proof of the asymptotic Theorem. $\diamondsuit$

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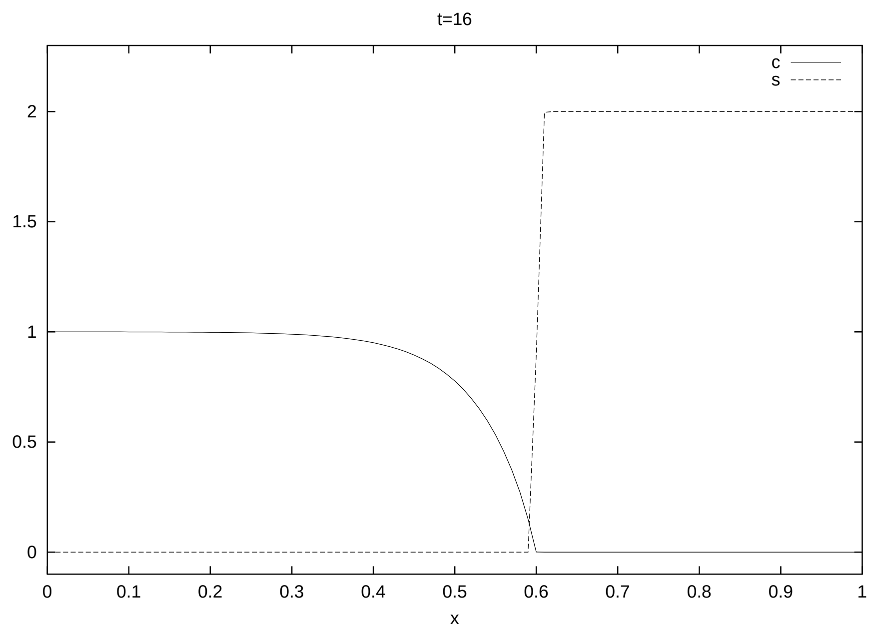

Figure 1: Simulation in $1D$ with $\phi(c) = \mu\, c$ and $k = 10^4$

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